Coupled chemotaxis fluid model

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Abstract

We consider a model system for the collective behaviour of oxygen-driven swimming bacteria in an aquatic fluid. In certain parameter regimes such suspensions of bacteria feature large-scale convection patterns as a result of the hydrodynamic interaction between bacteria. The presented model consist of a parabolic-parabolic chemotaxis system for the oxygen concentration and the bacteria density coupled to an incompressible Stokes equation for the fluid driven by a gravitational force of the heavier bacteria.

We show local existence of weak solutions in a bounded domain in $\mathbb{R}^d$, $d = 2, 3$ with no-flux boundary condition and in $\mathbb{R}^2$ in the case of inhomogeneous Dirichlet conditions for the oxygen.

1 Oxygen-driven swimming bacteria in an aquatic fluid

We consider a model for the collective behaviour of a suspension of oxygen-driven bacteria in an aquatic fluid.

As motivation we shall consider an experiment where swimming bacteria of the kind *Bacillus subtilis* are suspended in a drop of water confined within two (vertical and invisible) glass plates 1 mm apart (see Fig. 1; references [6,18] [1]). The bacteria suspension, occupying a volume fraction of about 1% and initially almost homogeneously distributed [Fig. 1 A], evolves as some bacteria swim upwards an oxygen gradient, while other bacteria run out of oxygen and are rendered immobile [Fig. 1 B, C]. The oxygen itself diffuses into the water through the water surface. Since the bacteria are about 10% denser than water instabilities develop at the high concentration layer close to the water surface [Fig. 1 D]. Bacteria-rich plumes form and start to move

[1] All pictures are taken from [6] and [18]

sideways along the curved surface. [Fig. 1 E]. Due to these large scale fluid motions formerly inactive bacteria will be reoxygenated and participate in the established large scale convection pattern. [Fig. 1 F, Fig. 2].

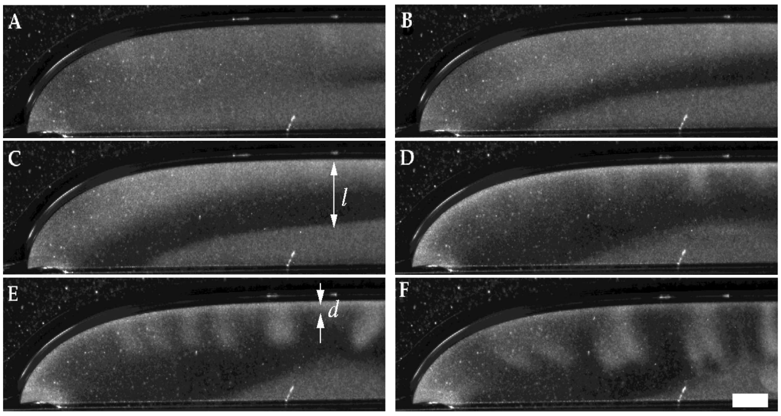

Figure 1: From uniformly distributed bacteria to a flow in the entire drop. The scale bar is 1 mm.

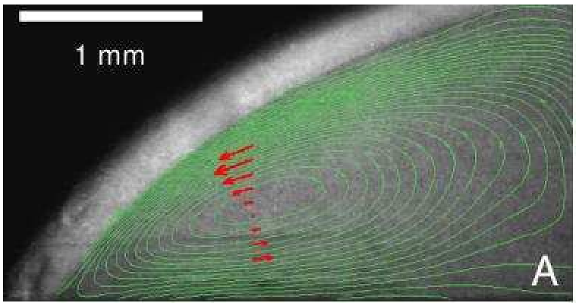

Figure 2: flow pattern

In [18] the authors proposed the following set of model equations:

$$\begin{cases} c_t + u \cdot \nabla c - D_c \, \Delta c = -n\kappa f(c) \\ n_t + u \cdot \nabla n - D_n \Delta n = -\chi \nabla \cdot [n r(c) \nabla c] \\ \rho(u_t + u \cdot \nabla u) + \nabla p - \eta \Delta u + n \nabla \Phi = 0 \\ \nabla \cdot u = 0 \end{cases} \tag{1}$$

where c and n denote the concentration of oxygen and bacteria, respectively, and u denotes the velocity field of the fluid subject to an incompressible Navier-Stokes-type equation with pressure p and viscosity η and a modelling the gravitational force $\nabla\Phi := V_b g \rho_{rel} \hat{z}$ exerted from a bacterium onto the fluid along the downwards unit vector $\hat{z}$ and proportional to the volume of the bacteria V_b, the gravitation acceleration $g = 9.8m/s^2$, and the density difference of the bacteria and water ρ_{rel} (bacteria are about 10% denser than water). Since the fluid motion is slow, we can also just use Stokes. It is assumed that the total contribution of the bacteria to the density of the bacteria suspension is small *i.e.* density of the bacteria suspension equals the density of water as used in the biophysics literature [6, 18].

Both, the oxygen concentration c and the density of the bacteria n are transported by the fluid and diffuse with their respective diffusion constants D_c and D_n. Moreover, the oxygen is consumed proportional to the density of cells n and a cut-off function $f(c)$, which models an inactivity threshold of the bacteria due to low oxygen supply. The bacteria are directed towards a higher oxygen gradient according a Keller-Segel type model with the chemotactic sensitivity χ and a second cutoff-function $r(c)$. We remark that in the present model the oxygen is consumed by the bacteria rather than produced as in the classical Keller-Segel model.

Experiments suggest that the cut-off functions can be modelled, for instance, by a stepfunction: $r(c) = f(c) := \theta(c - c^*)$.

For the system (1) the experimental setup corresponds to mixed boundary conditions: First, let us split the boundary into two parts:

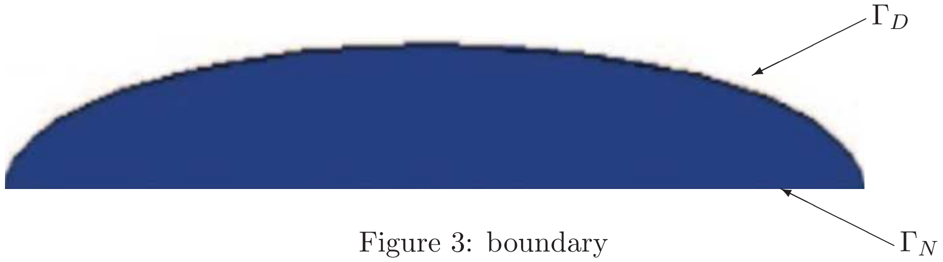

Figure 3: boundary

B. c. at the upper curved part Γ_D:

$$\nabla n \cdot \boldsymbol{\nu} = 0, \qquad c = c_{saturation}, \qquad u \cdot \boldsymbol{\nu} = 0, \qquad \frac{\partial^2}{\partial \nu^2} (u \cdot \boldsymbol{\nu}) = 0;$$

B. c. at the lower flat part Γ_N:

$$\nabla n \cdot \boldsymbol{\nu} = 0, \qquad \nabla c \cdot \boldsymbol{\nu} = 0, \qquad u = 0;$$

So we have no-flux of cells and oxygen at the bottom (and the sides) of the drop, no flux of cells through the fluid-air interface, no fluid flow through the fluid-air interface, whereas at the fluid-glass boundary the fluid velocity u is 0. The oxygen concentration outside the drop can be assumed equal to its saturation value inside the fluid.

In a large variety of ecological systems swimming micro-organisms play a pivotal role, be it plankton, the bottom of the ocean food chain, or algae and their impact on the CO_2 and O_2 balance influencing even the world climate. Related collective behaviour is observed as widely as in flocking of bird, swarming of animals, and even crowding of humans [9] [14].

Nevertheless, standard models for chemotaxis aggregation such as the Patlak-Keller-Segel model do neglect the surrounding fluid and would fail to predict the large scale bioconvection affecting clearly the overall oxygen consumption in the above experiments.

In the present paper, we show local existence of weak solutions for a coupled chemotaxis-fluid system.

Remark 1.1 *A good overview for chemotaxis can be found in [15]. For the elliptic-parabolic Keller-Segel model in the 2 D case, the results are summarized in [3] and for the parabolic-parabolic model recent progress has been achieved in [4]. Model hierarchies for cell aggregation by chemotaxis are described in [5]. A good reference for the Stokes equation is [10]. All we need is that its solution behaves almost like one of a parabolic equation.*

The rest of this paper is organized as follows: In section 2, we give a local existence result in $\mathbb{R}^3$ for the system (1) with the Navier-Stokes equation simplified to Stokes equation. In section 3, we prove local existence in $\mathbb{R}^3$ for the system (1) with the Navier-Stokes equation. In section 4, we work on an extended model and in section 5, we show local existence in $\mathbb{R}^2$ for the system (1) with the Navier-Stokes equation simplified to Stokes equation and mixed boundary data for c.

1.1 Notation

$|\cdot|$ denotes the $L^2(\Omega)$-norm, $|\cdot|_p$ the $L^p(\Omega)$-norm for $1 \le p \le \infty$. For l integer, $W_q^l(\Omega)$ is the Banach space consisting of all elements of $L^q(\Omega)$ having generalized derivatives up to order l inclusively that are q-th power summable on Ω. The norm in $W_q^l(\Omega)$ is defined by the equality

$$|v|_{W_q^l(\Omega)} := \left(\sum_{|\alpha| \le l} \int_\Omega |D^\alpha v|^q \, dx \right)^{1/q} .$$

For nonintegral l, let $\lfloor l \rfloor$ be the largest integer less than l. Then $W_q^l(\Omega)$ is defined as the Banach space consisting of all elements of $W_q^{\lfloor l \rfloor}(\Omega)$ with finite norm

$$|v|_{W_q^l(\Omega)} := |v|_{W_q^{\lfloor l \rfloor}(\Omega)} + \sum_{|\alpha| = \lfloor l \rfloor} \left(\int_\Omega \int_\Omega |D^\alpha v(x) - D^\alpha v(y)|^q \frac{dxdy}{|x-y|^{d+q(l-\lfloor l \rfloor)}} dx \right)^{1/q} .$$

Let Q_T be the parabolic cylinder $(0,T) \times \Omega$. For integral l, $W_q^{2l,l}(Q_T)$ is the Banach space consisting of all elements of $L^q(Q_T)$ having generalized derivatives of the form $D_t^r D_x^\alpha$ with $2r + |\alpha| \le 2l$ that are q-th power summable on Ω. The norm is defined by

$$|v|_{W_q^{2l,l}(\Omega)} := \left(\sum_{2r+|\alpha|\leq l} \int_\Omega |D_t^r D_x^\alpha v|^q \, dx \right)^{1/q}$$

2 Model with Stokes equation in 3 D

Let Ω be a bounded domain in $\mathbb{R}^3$ with smooth boundary. We seek a solution (c, n, u) of

$$\begin{cases} c_t + u \cdot \nabla c = \Delta c - n f(c) & (2) \\ n_t + u \cdot \nabla n = \Delta n - \nabla \cdot (n \chi \nabla c) & (3) \\ u_t + \nabla p - \eta \Delta u + n \nabla \Phi = 0 & (4) \\ \nabla \cdot u = 0 & (5) \end{cases}$$

together with the following boundary conditions:

$$\left.\frac{\partial c}{\partial \nu}\right|_{\partial\Omega \times [0,T]} = \left.\frac{\partial n}{\partial \nu}\right|_{\partial\Omega \times [0,T]} = 0 \text{ and } \left. u \right|_{\partial\Omega \times [0,T]} = 0 \qquad (6)$$

where η, χ are positive given constants; Φ time-independent, $\nabla \Phi \in L^\infty(\Omega)$; $f : \mathbb{R} \mapsto [0, \infty)$ is once continuously differentiable, monotonically increasing, $f(0) = 0$ and p is the Lagrangian multiplier associated to $\nabla \cdot u = 0$. The boundary conditions used in (6) are a simplification of the boundary conditions in the experiment. In the 2-dimensional case, the mixed boundary conditions for the c-equation are handled in section 5.

On the initial data we assume

$$0 \leq c_0 \in W_{8/3}^{2-2/(8/3)}(\Omega), 0 \leq n_0 \in L^3(\Omega) \text{ and } u(x, 0) = u_0(x) \in H_0^1(\Omega)$$
$$\text{with } \nabla \cdot u_0 = 0 \qquad (7)$$

Theorem 2.1 *There exists $T > 0$ such that the system (2) - (7) has a weak solution (c, n, u) in the sense of distributions with $0 \leq c \in W_{8/3}^{2,1}((0, T) \times \Omega)$, $0 \leq n \in L^\infty(0, T; L^3(\Omega)) \cap L^2(0, T; H^1(\Omega))$ and $u \in L^2(0, T; H^2(\Omega))$.*

Remark 2.2 *The same results can be shown with an additional cut-off function $r(c)$ in the n-equation but it makes the proof even more technical.*

For global existence, we would need good a-priori estimates. One way of finding these estimates which works very well for coupled convection-diffusion systems is by an entropy method.

Standard entropies like $\int n \ln(n)$ do not see the fluid but for the c-equation $|\nabla c|^2$ seems to be the correct term. Then this gives an additional term $(\Delta c) u \cdot \nabla c$ in the "entropy dissipation" and we cannot give it a sign *nor bound it by other terms. Neglecting the transport term in the c-equation is not realistic: When the fluid transports the bacteria, why should it not transport the chemical?*

Proof 1 Lemma 2.3 *For given* $u \in L^2(0,T;H^2(\Omega))$ *and* $0 \leq n \in L^3(0,T;L^3(\Omega))$*, with initial and boundary conditions given in the theorem 2.1, equation* (2) *has a solution in* $L^2((0,T)\times\Omega)$ *and* $0 \leq c \leq |c_0|_\infty$.

Proof 2 *Consider this equation*

$$c_t + u \cdot \nabla c = \Delta c - n f(\tilde{c}) \tag{8}$$

Let us define $c_M := |c_0|_\infty < \infty$*. We work in the space*
$X := \{\tilde{c} : \tilde{c} \in L^2((0,T)\times\Omega), 0 \leq \tilde{c} \leq c_M \ a.e.\}$

Let us take $\tilde{c}_1, \tilde{c}_2 \in X$ *and solve* (8) *for* c_1, c_2*. Then taking the difference of the two equations, multiplying them by* $(c_1 - c_2)$ *and using* $\nabla \cdot u = 0$*, we have*

$$\frac{d}{dt}|c_1 - c_2|_2^2 + 2|\nabla c_1 - \nabla c_2|_2^2 \leq -2\int_\Omega [f(\tilde{c}_1) - f(\tilde{c}_2)]n(c_1 - c_2)$$
$$\frac{d}{dt}|c_1 - c_2|_2^2 \leq C|c_0|_\infty \max_{0\leq c\leq c_M}(|f'(c)|)\int_\Omega |\tilde{c}_1 - \tilde{c}_2||n|$$

So we obtain

$$\begin{aligned}|c_1 - c_2|_2^2(t) &\leq C(\Omega)|c_0|_\infty \max_{0\leq c\leq c_M}(|f'(c)|)\int_0^t |\tilde{c}_1 - \tilde{c}_2|_2|n|_3 \\ &\leq C(\Omega)|c_0|_\infty \max_{0\leq c\leq c_M}(|f'(c)|)t^{1/6}|\tilde{c}_1 - \tilde{c}_2|_{L^2(Q_T)}|n|_{L^3(Q_T)}\end{aligned}$$

For t *small enough, we have a contraction. Moreover, multiplying* (8) *by* c_-*, integrating and using the mean-value-theorem, we obtain* $c \geq 0$ *a.e. Similarly, multiplying* (8) *by* $(c - |c_0|_\infty)_+$*, and integrating, we obtain* $c \leq |c_0|_\infty$ *a.e. So applying Banach fixed-point theorem gives us* c *which solve equation* (2) *on* $[0,t]$*. Iterating the method, we prove the existence of a solution of* (2) *on an arbitrary time interval* $[0,T]$.

Let us define $K := \max_{0\leq c\leq c_M}(|f(c)|)$.
We work in the space $Y := \{\tilde{n} : \tilde{n} \in L^3(0,T;L^3(\Omega)), \tilde{n} \geq 0 \ a.e.\}$.
Structure of the proof: We define a nonlinear operator $N : \tilde{n} \mapsto n, N : Y \to Y$*, in the following way*

1. *We take* $\tilde{n}$*, put it in (4) to obtain* u*,*
2. *obtain* c *from* (2) *with lemma 2.3,*
3. *then put* c*,* u *in (3) and solve the linear PDE for* n*.*

In order to apply Schauder's Fixed Point Theorem, we need to check: self-mapping, continuity and compactness.

2.1 Self-mapping

u-equation: *Put* $\tilde{n}$ *in (4). According to [10], there exists a solution* $u \in L^2(0,T;H^2(\Omega))$ *and this inequality holds:*

$$\int_0^T |u|^2_{H^2(\Omega)} + |u_t|^2 \; dt \leq C \left(|u(0)|^2_{H^1(\Omega)} + \int_0^T |\tilde{n}|_2^2 \right) \tag{9}$$

Remark 2.4 *Assuming different* L^p*-regularity on the initial data* u_0*, gives different regularity* u *i.e.* $u_0 \in W_p^{2-2/p}(\Omega)$*,* u_0 *vanishes on the boundary and* $\nabla \cdot u_0 = 0$ *implies* $u \in W_p^{2,1}(\Omega)$ *for* $1 < p < \infty$ *and*

$$\int_0^T |u|^p_{W_p^2(\Omega)} + |u_t|^p \; dt \leq C \left(|u(0)|^p_{W_p^1(\Omega)} + \int_0^T |\tilde{n}|_p^p \right)$$

There are also regularity results with mixed exponents, see e.g. [8], [7]. A good overview is provided by [13].

c-equation: *Consider the following: Since* $u \in L^2(0,T;H^2(\Omega))$ *and* $u_t \in L^2(0,T;L^2(\Omega))$ *follows* $u \in C(0,T;H^1(\Omega))$ *and* $u \in L^5((0,T)\times\Omega)$ *in* $\mathbb{R}^3$*.*

Theorem 2.5 *Since* $|u|_{L^5((0,T)\times\Omega)}$ *is finite and* $|u|_{L^5((t,t+\tau)\times\Omega)}$ *tends to zero for* $\tau \to 0$*,* $\tilde{n} \in L^{8/3}(Q_T)$*,* $c_0 \in W^{2-2/(8/3)}_{(8/3)}(\Omega)$*, the regularity of* c *given by lemma 2.3 can be improved: equation (2) has a unique solution* $c \in W^{2,1}_{(8/3)}(Q_T)$ *and we have the following estimate*

$$|c|_{W^{2,1}_{(8/3)}(Q_T)} \leq C(|u|_{L^5((0,T)\times\Omega)}) \left(K|\tilde{n}|_{L^3((0,T)\times\Omega)} + |c_0|_{W^{2-2/(8/3)}_{(8/3)}(\Omega)} \right) \tag{10}$$

Moreover, the constant C *depends continuously on* $|u|_{L^5((0,T)\times\Omega)}$*.*

Proof 3 *[11] Chapter IV, §9, Theorem 9.1, p. 341 and Remark p. 351. They first solve the heat equation and then consider perturbations of the heat equation.*

From Gagliardo-Nirenberg-Sobolev inequality in 3 D $|v(t)|_{6+\delta} \leq C|v(t)|^{1-\alpha}_{8/3}|\nabla v(t)|^{\alpha}_{8/3}$ *for* $\delta \leq 1/4$ *and* $\alpha = (3/8 - 4/25)3 = 129/200$*, we obtain:*

$$|\nabla c|_{L^{4+\delta_1}(0,T;L^{6+\delta}(\Omega))} \leq C\left[|\nabla c|_{L^\infty(0,T;L^{8/3}(\Omega))} + |D^2 c|_{L^{8/3}((0,T)\times\Omega)} \right] \leq C|c|_{W^{2,1}_{8/3}(\Omega)} \tag{11}$$

with $\delta_1 \leq 13/100$

n-equation: *Multiply* (3) *by* n^{p-1} *and integrate. As* (5) *holds and* $|n|_3 \le C|n|^{1/2}_{H^1(\Omega)}|n|^{1/2}_2$, *we have:*

$$\frac{d}{dt}|n|_p^p = p\int n^{p-1}n_t = -p\int(\nabla n^{p-1})(\nabla n - \chi n\nabla c)$$

$$\frac{d}{dt}|n|_p^p(t) + \frac{4(p-1)}{p}|\nabla n^{p/2}|^2_{L^2(\Omega)} \le C\int_\Omega n^{p/2}\nabla n^{p/2}\nabla c \le C|n^{p/2}|_3|\nabla n^{p/2}|_2|\nabla c|_6$$

using $|n|_3 \le C|n|^{1/2}_{H^1(\Omega)}|n|^{1/2}_2$ *and for the next inequality* $ab \le \epsilon a^{4/3} + Cb^4$

$$\le C\left|n^{p/2}\right|^{1/2}_{H^1(\Omega)}\left|n^{p/2}\right|^{1/2}_2\left|\nabla n^{p/2}\right|_2|\nabla c|_6 \le \frac{2(p-1)}{p}\left|n^{p/2}\right|^2_{H^1(\Omega)} + C|n|_p^p|\nabla c|_6^4$$

$$\Rightarrow \frac{d}{dt}|n|_p^p + \frac{2(p-1)}{p}|\nabla n^{p/2}|_2^2 \le \frac{2(p-1)}{p}|n^{p/2}|_2^2 + C|n|_p^p|\nabla c|_6^4$$

Using Gronwall estimate, we obtain

$$|n(t)|_p^p \le |n_0|_p^p \exp\left(C\int_0^t\left[1+|\nabla c|_6^4\right]\right)$$

$$\le |n_0|_p^p \exp\left(Ct + C\left(K|\tilde{n}|_{L^3(Q_t)} + |c_0|_{W^{5/4}_{8/3}(\Omega)}\right)^4\right) \tag{12}$$

$$n^{p/2} \in L^2(0,T;H^1(\Omega)) \tag{13}$$

For given u and c in the spaces stated in the theorem 2.1, $n \in L^\infty(0,T;L^3(\Omega)) \cap L^2(0,T;H^1(\Omega))$ *is the unique solution of* (3). *Therefore, solving the equation*

$$\hat{n}_t + u\cdot\nabla\hat{n} = \Delta\hat{n} - \nabla\cdot(\hat{n}_+\chi\nabla c) \tag{14}$$

and multiplying it by $\hat{n}_-$, *we obtain* $\hat{n} \ge 0$. *So* $n \ge 0$.

$$|n|_{L^p(Q_T)} \le |n_0|_p T^{1/p}\exp\left(C'T + C'\left(K|\tilde{n}|_{L^3(Q_T)} + |c_0|_{W^{5/4}_{8/3}(\Omega)}\right)^4\right) \tag{15}$$

So for R large enough and T small enough e.g. for $R := 2|n_0|_p$ *and* $T := \min\left(\exp(-2pK_1), K_1/C'\right)$ *where*

$$K_1 := C'\left(KR + |c_0|_{W^{5/4}_{8/3}(\Omega)}\right)^4$$

the ball $B_2 := \{\tilde{n} \in Y : |\tilde{n}|_{L^3(Q_T)} \le R\}$ *is invariant under* N.

2.2 Continuity

u-equation: *(4) is a linear PDE with $n\nabla\Phi$ as an inhomogeneity, so when we take $\tilde{n}_1$ and $\tilde{n}_2$, solve (4) for u_1 and u_2, we obtain (cf.* (9)*)*

$$\int_0^T |u_1 - u_2|_\infty^2 \leq C \int_0^T |\tilde{n}_1 - \tilde{n}_2|_2^2 \tag{16}$$

c-equation: *Let $\tilde{c}_1$ and $\tilde{c}_2$ be the solution to* (2) *with $u = u_1$, $u = u_2$. Subtracting the two equations, we find using $\nabla \cdot u = 0$*

$$(c_1 - c_2)_t + \nabla \cdot [c_1 u_1 - c_2 u_2] = \nabla \cdot (\nabla c_1 - \nabla c_2) - (f(\tilde{c}_1)\tilde{n}_1 - f(\tilde{c}_2)\tilde{n}_2)$$

Multiplying it by $c_1 - c_2$ and integrating by parts, we obtain:

$$\begin{aligned} &\frac{d}{dt}|c_1 - c_2|_2^2 + 2|\nabla(c_1 - c_2)|_2^2 \leq 2\int_\Omega (c_1 - c_2)\nabla[c_2 u_1 - c_2 u_2] \\ &\quad - 2\int_\Omega [f(c_1)\tilde{n}_1 - f(c_1)\tilde{n}_2 + f(c_1)\tilde{n}_2 - f(c_2)\tilde{n}_2](c_1 - c_2) \\ &\quad \textit{using the monotonicity of } f \\ &\qquad \leq |\nabla(c_1 - c_2)|_2^2 + |u_1 - u_2|_\infty^2 |c_2|_2^2 + C|c_1 - c_2|_2^2 + |\tilde{n}_1 - \tilde{n}_2|_2^2 \end{aligned}$$

For a sequence $\tilde{n}_k \to \tilde{n}$ in $L^3((0,T) \times \Omega)$, let us define the gradient $g_k := \nabla c(\tilde{n}_k)$ and $g := \nabla c(\tilde{n})$. From the differential inequality just derived, we obtain:

$$\lim_{k\to\infty} |g_k - g|_{L^2(0,T;L^2(\Omega))} = 0 \tag{17}$$

Because of (11) *and the bound* (10)*, we find a subsequence g_k which converges in $L^4(0,T;L^6(\Omega))$. Since the limit has to be g, $g \in L^4(0,T;L^6(\Omega))$ and since the limit is unique, we have*

$$\nabla c(\tilde{n}_k) \to \nabla c(\tilde{n}) \text{ in } L^4(0,T;L^6(\Omega)). \tag{18}$$

n-equation: *Let $\tilde{n}_1$ and $\tilde{n}_2$ be given. Infer $n = \tilde{n}_1$ in (4). Solve (4) for u_1. Infer $u = u_1$ and $n = \tilde{n}_1$ in (2). Obtain c_1 from equation (2). Finally solve (3) for n_1 with $u = u_1$ and $c = c_1$. Analogously, we obtain n_2. By subtracting the two equations we have:*

$$(n_1 - n_2)_t + \nabla[n_1 u_1 - n_2 u_2] = \nabla \cdot (\nabla n_1 - \nabla n_2) - \chi\nabla \cdot [n_1 \nabla c_1 - n_2 \nabla c_2] \tag{19}$$

We multiply this equation by $p(n_1 - n_2)_+^{p-1}$ and integrate over Ω.

We consider the terms one after the other:

1. *"$\nabla[n_1u_1 - n_2u_2]$":*

$$\int p(n_1-n_2)_+^{p-1}\nabla[n_1u_1-n_2u_2] =$$

$$\underbrace{p\int(n_1-n_2)_+^{p-1}\nabla[n_1u_1-n_2u_1]}_{=\,0,\ since\ div(u_1)=0} + \int p(n_1-n_2)_+^{p-1}\nabla[n_2u_1-n_2u_2]$$

$$\le 2(p-1)\left|\int n_2(n_1-n_2)_+^{p/2-1}\nabla(n_1-n_2)_+^{p/2}[u_1-u_2]\right|$$

applying Hölder with the following exponents: $p, \dfrac{2p}{p-2}, 2, \infty$ *we obtain:*

$$\le 2(p-1)\left[\int(n_1-n_2)_+^{p}\right]^{\frac{p-2}{2p}}\left|\nabla(n_1-n_2)_+^{p/2}\right|_2|n_2|_p|u_1-u_2|_\infty \tag{20}$$

2. *"$-\chi\nabla\cdot[n_1\nabla c_1 - n_2\nabla c_2]$":*

$$-p\chi\int(n_1-n_2)_+^{p-1}\nabla\cdot[n_1\nabla c_1-n_2\nabla c_2] =$$

$$2\chi(p-1)\int(n_1-n_2)_+^{p/2-1}\nabla(n_1-n_2)_+^{p/2}\cdot[n_1\nabla c_1-n_2\nabla c_2] =$$

$$2\chi(p-1)\int(n_1-n_2)_+^{p/2-1}\nabla(n_1-n_2)_+^{p/2}\cdot[(n_1-n_2)\nabla c_1+n_2(\nabla c_1-\nabla c_2)] =$$

$$2\chi(p-1)\int(n_1-n_2)_+^{p/2}\nabla(n_1-n_2)_+^{p/2}\nabla c_1$$

$$+\,2\chi(p-1)\int n_2(n_1-n_2)_+^{p/2-1}\nabla(n_1-n_2)_+^{p/2}(\nabla c_1-\nabla c_2)$$

By Hölder inequality

$$\le (I)+(II) \tag{21}$$

with $(I) := 2\chi(p-1)\left|(n_1-n_2)_+^{p/2}\right|_3\left|\nabla(n_1-n_2)^{p/2}\right|_2|\nabla c_1|_6,$

$$(II) := 2\chi(p-1)\,|n_2|_{3p/2}\left(\int(n_1-n_2)_+^{3p/2}\right)^{\frac{p-2}{3p}}|\nabla(n_1-n_2)^{p/2}|_2|\nabla(c_2-c_1)|_6$$

For $p=3$ *Gagliardo-Nirenberg-Sobolev inequality* $|n^{3/2}|_3 \le C\left|n^{3/2}\right|_2^{1/2}\left|n^{3/2}\right|_{H^1(\Omega)}^{1/2}$ *gives an estimate of (I):*

$$(I) \le C\left|(n_1-n_2)_+^{3/2}\right|_2^{1/2}\left|(n_1-n_2)_+^{3/2}\right|_{H^1(\Omega)}^{1/2}\left|\nabla(n_1-n_2)_+^{3/2}\right|_2|\nabla c_1|_6$$

$$\le C\left|(n_1-n_2)_+^{3/2}\right|_2^{1/2}\left|(n_1-n_2)_+^{3/2}\right|_{H^1(\Omega)}^{3/2}|\nabla c_1|_6$$

Using $ab \leq \epsilon a^{4/3} + C(\epsilon) b^4$ *we obtain*

$$(I) \leq \frac{2}{3}\left(\left|\nabla(n_1-n_2)_+^{3/2}\right|_2^2 + \left|(n_1-n_2)_+^{3/2}\right|_2^2\right) + C\left|(n_1-n_2)_+^{3/2}\right|_2^2 |\nabla c_1|_6^4$$

Similarly for $p=3$ *Gagliardo-Nirenberg-Sobolev inequality* $|n|_3 \leq C\,|n|_2^{1/2}\,|n|_{H^1(\Omega)}^{1/2}$ *and the boundedness of* $\left|n_2^{3/2}(t)\right|_2$ *give an estimate of (II):*

$$\begin{aligned}(II) \leq C\left|n_2^{3/2}\right|_2^{1/3}\left|n_2^{3/2}\right|_{H^1(\Omega)}^{1/3}\left|(n_1-n_2)_+^{3/2}\right|_2^{1/6} \\ \cdot\left|(n_1-n_2)_+^{3/2}\right|_{H^1(\Omega)}^{1/6}\left|\nabla(n_1-n_2)^{3/2}\right|_2 |\nabla(c_2-c_1)|_6 \\ \leq C\left|n_2^{3/2}\right|_{H^1(\Omega)}^{1/3}\left|(n_1-n_2)_+^{3/2}\right|_2^{1/6}\left|(n_1-n_2)_+^{3/2}\right|_{H^1(\Omega)}^{7/6} |\nabla(c_2-c_1)|_6\end{aligned}$$

Using $ab \leq \epsilon a^{12/7} + C(\epsilon) b^{12/5}$ *we obtain*

$$(II) \leq \frac{2}{3}\left(\left|\nabla(n_1-n_2)_+^{3/2}\right|_2^2 + \left|(n_1-n_2)_+^{3/2}\right|_2^2\right) + C\left|n_2^{3/2}\right|_{H^1(\Omega)}^{1/3\cdot 12/5} |\nabla(c_2-c_1)|_6^{12/5}$$

Let us now do the computation for the full equation: Since

$$\begin{aligned}\int p(n_1-n_2)_+^{p-1}\left[(n_1-n_2)_t - \Delta(n_1-n_2)\right] \\ = \frac{d}{dt}\int (n_1-n_2)_+^p + \frac{4(p-1)}{p}\int\left(\nabla(n_1-n_2)_+^{p/2}\right)^2\end{aligned}$$

using (19), (20), (21) with $p=3$ *and cancelling the gradient term, we obtain:*

$$\begin{aligned}\frac{d}{dt}|(n_1-n_2)_+|_3^3 + \frac{2}{3}\left|\nabla(n_1-n_2)_+^{3/2}\right|_2^2 \\ \leq 4|n_2|_3\left(\int (n_1-n_2)_+^3\right)^{1/6}\left|\nabla(n_1-n_2)_+^{3/2}\right|_2 |u_1-u_2|_\infty \\ + \frac{2}{3}\,|(n_1-n_2)_+|_3^3 + C\,|(n_1-n_2)_+|_3^3\,|\nabla c_1|_6^4 \\ + \frac{2}{3}\,|(n_1-n_2)_+|_3^3 + C\left|n_2^{3/2}\right|_{H^1(\Omega)}^{4/5} |\nabla(c_2-c_1)|_6^{12/5}\end{aligned}$$

Using $ab \leq \epsilon a^2 + C(\epsilon) b^2$ *and the boundedness of* $|n_1(t)|_3\,, |n_2(t)|_3$ *, we obtain*

$$\frac{d}{dt} g(t) \leq \alpha(t) + \beta(t) f(t) + \gamma(t)$$

where $g(t) := |(n_1 - n_2)_+|_3^3$, $\alpha(t) := C \left|\nabla(n_1 - n_2)_+^{3/2}\right|_2 |u_1 - u_2|_\infty$, $\beta(t) := \frac{4}{3} + C|\nabla c_1|_6^4$ *and* $\gamma(t) := C \left|n_2^{3/2}\right|_{H^1(\Omega)}^{4/5} |\nabla(c_2 - c_1)|_6^{12/5}$. *Since*

$$\int_0^T \gamma(t) \leq \left(\int_0^T \left|n_2^{3/2}\right|_{H^1(\Omega)}^{4/5 \cdot 5/2}\right)^{2/5} \left(\int_0^T |\nabla(c_2 - c_1)|_6^{12/5 \cdot 5/3}\right)^{3/5}$$

using (16) and (18), we obtain continuity from the differential equation above.

2.3 Compactness

This compactness argument was done for a simpler system in [2], it relies on the Aubin-Lions compactness lemma, (see [12], Ch. IV, 4 and [1], and [16]). A simple statement goes as follows:

Lemma 2.6 (Aubin-Lions compactness lemma) *Let* H *be a Banach space and* V *be compactly embedded in* H *and take* $T > 0$, $p \in (1, \infty)$. *Let* $(w_k)_{k\in\mathbb{N}}$ *be a bounded sequence of function in* $L^p(0, T; V)$ *with* $(\dot{w})_k$ *bounded in* $L^p(0, T; V')$ *uniformly with respect to* $k \in \mathbb{N}$, *then* $(w_k)_{k\in\mathbb{N}}$ *is relatively compact in* $L^p(0, T; H)$.

Apply Aubin-Lions lemma to $H = L^2(\Omega), V = H^1(\Omega)$ *which are such that:*

$$H^1(\Omega) \overset{\text{compact}}{\hookrightarrow} L^2(\Omega) \overset{\text{continuous}}{\hookrightarrow} H^{-1}(\Omega)$$

Since $n_t \in L^2(0, T; H^{-1}(\Omega))$, *we obtain:* S *defined as the closure of* $N(\{\tilde{n} : |\tilde{n}|_{L^3((0,T)\times\Omega)} \leq R\})$ *is compact in* $L^2((0, T) \times \Omega)$. *Take a sequence* $(w)_k$ *in* S, *then there is a subsequence* $w_k \longrightarrow w$ *in* $L^2((0, T) \times \Omega)$*-norm. Since* w_k *is bounded in* $L^\infty(0, T; L^3(\Omega))$, w_k *converges against* w *in* $L^3(0, T; L^3(\Omega))$.

3 Extended model

We remark that it could be more realistic to include both the impact of gravity (potential force) on the bacteria and the effect of the chemotactic force on the fluid. This leads to the extended model system

$$\begin{cases} c_t + u \cdot \nabla c = \Delta c - nf(c) & (22) \\ n_t + u \cdot \nabla n = \Delta n - \nabla \cdot (n\chi\nabla c) + \nabla \cdot (n\nabla\Phi) & (23) \\ u_t + \nabla p - \eta\Delta u + n\nabla\Phi - n\chi(c)\nabla c = 0 & (24) \\ \nabla \cdot u = 0 & (25) \end{cases}$$

Theorem 3.1 *There exists $T > 0$ such that the system (22) - (25) has a weak solution (c, n, u) in the sense of distributions with $0 \leq c \in W^{2,1}_{8/3}((0,T)\times \Omega)$,*
$0 \leq n \in L^\infty(0,T; L^3(\Omega)) \cap L^2(0,T; H^1(\Omega))$ and $u \in L^2(0,T; H^2(\Omega))$.

Proof 4 *We will only give the main differences:*

Define $Z_3 := \{\tilde{n} : \tilde{n} \in L^4(0,T; L^3(\Omega)), \tilde{n} \geq 0 \ a.e.\} \times L^4(0,T; W^1_6(\Omega))$ and the nonlinear operator $A_3(\tilde{n}, \tilde{c}) : Z \to Z$ by

1. *We take $\tilde{n}$ and $\tilde{c}$, put it in (24) to obtain u,*
2. *obtain c from (22) with lemma 2.3,*
3. *then put c, u in (23) and solve the linear PDE for n.*

Since $\int_0^T |\tilde{n}\nabla\tilde{c}|_2^2 \leq \int_0^T |\tilde{n}|_3^2 |\nabla\tilde{c}|_6^2 \leq |\tilde{n}|^2_{L^4(0,T;L^3(\Omega))} |\nabla\tilde{c}|^2_{L^4(0,T;L^6(\Omega))}$, according to [10], there exists a solution $u \in L^2(0,T; L^2(\Omega))$ and this inequality holds:

$$\int_0^T |u|^2_{H^2(\Omega)} + |u_t|^2 \ dt \leq$$
$$C\left(|u(0)|^2_{H^1(\Omega)} + \int_0^T |\tilde{n}|_2^2 \, dt + |\tilde{n}|^2_{L^4(0,T;L^3(\Omega))} |\nabla\tilde{c}|^2_{L^4(0,T;L^6(\Omega))}\right) \quad (26)$$

So similar to section 2, for R large enough and T small enough, the ball $B_3 := \{(\tilde{n}, \tilde{c}) \in Z_3 : |\tilde{n}|_{L^4(0,T;L^3(\Omega))} + |\tilde{c}|_{L^4(0,T;W^1_6(\Omega))} \leq R\}$ is invariant under A_3.

We take $\tilde{n}_1$, $\tilde{n}_2$ and $\tilde{c}_1$, $\tilde{c}_2$, solve (24) for u_1 and u_2, take their difference:

$$\int_0^T |u_1 - u_2|^2_{H^2(\Omega)} \ dt \leq C \int_0^T |\tilde{n}_1 - \tilde{n}_2|_2^2 \, dt$$
$$+ C|\tilde{n}_1 - \tilde{n}_2|^2_{L^4(0,T;L^3(\Omega))} |\nabla\tilde{c}_1 - \nabla\tilde{c}_2|^2_{L^4(0,T;L^6(\Omega))} \quad (27)$$

Therefore together with the continuity results from section 2, $A_3 : B_3 \to B_3$ is continuous. The compactness can also be obtained similarly to section 2. So the Schauder fixed-point theorem gives us a solution (c, n, u).

4 Model with Navier-Stokes

Since the fluid flow in the experiment is slow, the Navier-Stokes equation can be simplified to the Stokes equation. However here for completeness, we give a local existence result for the model with the Navier-Stokes equation.

$$\begin{cases} c_t + u \cdot \nabla c = \Delta c - n f(c) & (28) \\ n_t + u \cdot \nabla n = \Delta n - \nabla \cdot (n \chi \nabla c) & (29) \\ u_t + u \cdot \nabla u + \nabla p - \eta \Delta u + n \nabla \Phi = 0 & (30) \\ \nabla \cdot u = 0 & (31) \end{cases}$$

Theorem 4.1 *There exists $T > 0$ such that the system (28) - (31) has a weak solution (c, n, u) in the sense of distributions with $0 \leq c \in W^{2,1}_{8/3}((0,T) \times \Omega)$,*
$0 \leq n \in L^\infty(0,T; L^3(\Omega)) \cap L^2(0,T; H^1(\Omega))$ and $u \in L^2(0,T; H^2(\Omega))$.

Proof 5 *We will only give the main differences:*
Define $Z_4 := \{\tilde{n} : \tilde{n} \in L^\infty(0,T; L^3(\Omega)), \tilde{n} \geq 0$ a.e.$\}$ and a nonlinear operator: $A_4 : Z_4 \to Z_4$ by

1. *We take $\tilde{n}$, put it in (30) to obtain u,*
2. *obtain c from (28) with lemma 2.3,*
3. *then put c, u in (29) and solve the linear PDE for n.*

Since $\tilde{n} \in L^\infty(0,T; L^2(\Omega))$, we obtain $u \in L^\infty(0,T; H^1(\Omega)) \cap L^2(0,T; H^2(\Omega))$ and $u_t \in L^2(0,T; L^2(\Omega))$ according to [17]. So similar to section 2, for R large enough and T small enough, the ball $B_4 := \{\tilde{n} \in Z_4 : |\tilde{n}|_{L^\infty(0,T;L^3(\Omega))} \leq R\}$ is invariant under A_4.

Define

$$b(u, v, w) := \sum_{i,j=1}^{3} \int_\Omega u_i \partial_i v_j w_j \tag{32}$$

and recall for $u \in H_1(\Omega)$ with $\nabla \cdot u = 0$ we have

$$b(u, v, v) = 0 \text{ and } b(u, v, w) = -b(u, w, v) \quad \forall v, w \in H^1_0(\Omega) \tag{33}$$

We take $\tilde{n}_1$ and $\tilde{n}_2$, solve (30) for u_1 and u_2, take their difference, multiply it by $u := u_1 - u_2$ and integrate:

$$\frac{d}{dt}|u|^2 + 2|\nabla u|^2 = 2b(u_2, u_2, u) - 2b(u_1, u_1, u) + \int_\Omega (n_2 \nabla\Phi - n_1 \nabla\Phi) \cdot u$$

Using (33) and $|v|_4 \leq \sqrt{2}|v|^{1/4}|\nabla v|^{3/4}$ for $v \in H^1_0(\Omega)$, we obtain

$$\begin{aligned}\frac{d}{dt}|u|^2 + 2|\nabla u|^2 &= -2b(u, u_2, u) + \int_\Omega (n_2 - n_1)\nabla\Phi \cdot u \\ &= 2b(u, u, u_2) + \int_\Omega (n_2 - n_1)\nabla\Phi \cdot u\end{aligned}$$

Since $2b(u, u, u_2) \leq 2\sqrt{2}|u|^{1/4}|\nabla u|^{7/4}|u_2|_4 \leq 1/2|u|^2 + C|u|^2|u_2|_4^8$, we have

$$\frac{d}{dt}|u|^2 + 2|\nabla u|^2 \leq |u|^2 + C|u|^2|u_2|_4^8 + C|n_2 - n_1|_2^2$$

So we obtain

$$|u_1 - u_2|^2 = |u|^2 \le \exp\left(C\int_0^T |u_2|_4^8\,ds\right)\left(C\int_0^T |n_2 - n_1|_2^2\,ds\right)$$

$$|u_1 - u_2| \le \exp\left(C\int_0^T |u_2|_4^8\,ds\right)\left(CT|n_2 - n_1|_{L^\infty(0,T,L^3(\Omega))}\right)$$

Moreover, since $|u|_\infty \le C|\nabla u|^{1/2}|D^2u|^{1/2}$, *we have*

$$\int_0^T |u_1 - u_2|_\infty^2 \le \int_0^T |\nabla u||D^2u| \le |\nabla u|_{L^2(Q_T)}|D^2u|_{L^2(Q_T)}$$

$|D^2u|_{L^2(Q_T)}$ *is bounded and* $|\nabla u|_{L^2(Q_T)}$ *tends to* 0 *as* $|n_2 - n_1|_{L^\infty(0,T,L^3(\Omega))}$ *tends to* 0. *Therefore together with the continuity results from section 2,* $A_4 : B_4 \to B_4$ *is a contraction for* T *sufficiently small. So the Banach fixed-point theorem gives us a solution* (c, n, u).

5 2D mixed boundary data

Let Ω be a bounded domain in $\mathbb{R}^2$ with smooth boundary. We seek a solution (c, n, u) of

$$\begin{cases} c_t + u\cdot\nabla c = \Delta c - nf(c) & (34)\\ n_t + u\cdot\nabla n = \Delta n - \nabla\cdot(n\chi\nabla c) & (35)\\ u_t + \nabla p - \eta\Delta u + n\nabla\Phi = 0 & (36)\\ \nabla\cdot u = 0 & (37)\end{cases}$$

with these boundary conditions:

$$\frac{\partial c}{\partial\nu} = \frac{\partial n}{\partial\nu} = 0 \text{ on } \Gamma_N;\ c = K_c \text{ and } \frac{\partial n}{\partial\nu} - n\chi\frac{\partial c}{\partial\nu} = 0 \text{ on } \Gamma_D;\ u = 0 \text{ on } \partial\Omega \tag{38}$$

$K_c = const > 0$
and initial conditions:

$$0 \le n_0 \in L^2(\Omega), 0 \le c_0 \in H^1(\Omega)\cap L^\infty(\Omega), u_0(x) \in H_0^1(\Omega) \text{ with } \nabla\cdot u_0 = 0 \tag{39}$$

Theorem 5.1 *There exists* $T > 0$ *such that the system (34) - (39) has a weak solution* (c, n, u) *in the sense of distributions with* $c \in W_2^{2,1}((0,T)\times\Omega)$, $n \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$ *and* $u \in L^2(0,T;H^2(\Omega))$.

Remark 5.2 *When you increase the regularity assumption on the initial data* c_0 *and* n_0 *only a little bit i.e.* $n_0, \nabla c_0 \in L^{2+\epsilon}(\Omega)$, *you can put in the cut-off functions* r.

Proof 6 *The proof is simpler than in the 3 D case because in 2D the Gagliardo-Nirenberg-Sobolev inequality is better i.e.* $|n|_4 \le C|n|_2^{1/2}|\nabla n|_2^{1/2}$. *Therefore, I will only give the essential steps:*
We work in the space $Z_5 := \{\tilde{n} \in L^2(0,T;L^2(\Omega)) : \tilde{n} \ge 0\}$.

5.1 Self-mapping

c-equation:

$$\int c_t \Delta c + \int (u \cdot \nabla c)\Delta c = \int (\Delta c)^2 - \int f(c)\tilde{n}\Delta c$$
$$\frac{1}{2}\frac{d}{dt}|\nabla c|_2^2 - \int_{\Gamma_D} c_t \nabla c \cdot \nu + |\Delta c|_2^2 \le \frac{1}{4}|\Delta c|_2^2 + |u \cdot \nabla c|_2^2 + \frac{1}{4}|\Delta c|_2^2 + K^2|\tilde{n}|_2^2$$
$$\frac{1}{2}\frac{d}{dt}|\nabla c|_2^2 + \frac{1}{2}|\Delta c|_2^2 \le |u|_\infty^2 |\nabla c|_2^2 + K^2|\tilde{n}|_2^2$$

From this differential inequality, we obtain using (9):

$$|\nabla c(t)|_2^2 \le \exp\left(\eta|\nabla u(0)|_{L^2(\Omega)}^2 + C\int_0^t |\tilde{n}|_{L^2(\Omega)}^2\right)\left[|\nabla c_0|_2^2 + \int_0^t K^2|\tilde{n}|_2^2\right]$$

$$\nabla c \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)) \tag{40}$$

n-equation:

$$\frac{d}{dt}|n|_2^2 = 2\int n n_t = -2\int (\nabla n)(\nabla n - \chi n \nabla c)$$
$$\frac{d}{dt}|n|_2^2 + 2|\nabla n|_{L^2}^2 \le 2\chi|n\nabla n \nabla c|_1$$
$$\le 2\chi|\nabla n|_2|n|_4|\nabla c|_4 \le C|\nabla n|_2|n|_{H^1(\Omega)}^{1/2}|n|_2^{1/2}|\nabla c|_4$$
$$\le |n|_{H^1(\Omega)}^2 + C|n|_2^2|\nabla c|_4^4$$

So we obtain

$$\frac{d}{dt}|n|_2^2 + |\nabla n|_{L^2}^2 \le |n|_2^2 + C|n|_2^2|\nabla c|_4^4$$

$$|n(t)|_2^2 \le |n_0|_2^2 \exp\left(\int_0^t [1 + C(\chi)|\nabla c|_4^4]\right)$$

Using

$$|v|_{L^4(0,T;L^4(\Omega))} \le C\left[|v|_{L^\infty(0,T;L^2(\Omega))} + |\nabla v|_{L^2(0,T;L^2(\Omega))}\right] \tag{41}$$

we obtain

$$n \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)) \tag{42}$$

So for R *large enough and* T *small enough, the ball* $\{\tilde{n} \in Z_5 : |\tilde{n}|_{L^2(0,T;L^2(\Omega))} \le R\}$ *is invariant under* N.

5.2 Continuity

c-equation: *Consider*

$$(c_1 - c_2)_t + \nabla \cdot [c_1 u_1 - c_2 u_2] = \nabla \cdot (\nabla c_1 - \nabla c_2) - (f(c_1)\tilde{n}_1 - f(c_2)\tilde{n}_2)$$

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}|\nabla(c_1 - c_2)_+|_2^2 - \int_{\Gamma_D} (c_1 - c_2)_t \nabla(c_1 - c_2)_+ \cdot \nu + |\Delta(c_1 - c_2)_+|_2^2 \\
\le |\Delta(c_1 - c_2)_+ \nabla[c_1 u_1 - c_2 u_1]|_1 + |\Delta(c_1 - c_2)_+ \nabla[c_2 u_1 - c_2 u_2]|_1 \\
+ \frac{1}{4}|\Delta(c_1 - c_2)_+|_2^2 + C|\tilde{n}_1 - \tilde{n}_2|_2^2 \\
\le \frac{3}{4}|\Delta(c_1 - c_2)_+|_2^2 + C|u_1|_\infty^2 |\nabla(c_1 - c_2)_+|_2^2 + C|u_1 - u_2|_\infty^2 |\nabla c_2|_2^2 + C|\tilde{n}_1 - \tilde{n}_2|_2^2
\end{aligned}$$

n-equation: *Take $p = 2$ in the calculation done in the 3 D case.*

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